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Evaluating Fractional Fourier Series of Two Types of Matrix Fractional Functions Based on a New Multiplication of Fractional Analytic Functions

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we use some techniques to obtain the fractional Fourier series expansions of two types of matrix fractional functions. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of the results in ordinary calculus.**

Keyword: **New multiplication, fractional analytic functions, fractional Fourier series expansions, matrix fractional functions.**

I. INTRODUCTION

The history of fractional calculus is almost as long as the development of traditional calculus. In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field, such as Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, and Weyl. In the past few decades, fractional calculus has played a very important role in physics, electrical engineering, economics, biology, control theory, and other fields [1-15].

However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [16-20].

In this paper, based on a new multiplication of fractional analytic functions, we use some methods to find the fractional Fourier series expansions of following two types of matrix fractional functions:

$$E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha cos_\alpha\big(sin_\alpha(tAx^\alpha)\big),$$

$$E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha sin_\alpha\big(sin_\alpha(tAx^\alpha)\big),$$

where $0<\alpha\leq 1$, t is a real number, and A is a matrix. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, the definition of fractional analytic function is introduced.

Definition 2.1 ([21])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on

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closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.2 ([22])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{1}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\ . \tag{2}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes_\alpha\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{3}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}\otimes_\alpha\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}.\end{aligned} \tag{4}$$

Definition 2.3 ([23])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}. \tag{6}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha k}, \tag{7}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha k}. \tag{8}$$

Definition 2.4 ([24])**:** If $0<\alpha\leq 1$, x is a real variable and A is a matrix. The matrix α-fractional exponential function, matrix α-fractional cosine function, and matrix α-fractional sine function are defined as follows:

$$E_\alpha(Ax^\alpha)=\sum_{k=0}^{\infty}A^k\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}, \tag{9}$$

$$cos_\alpha(Ax^\alpha)=\sum_{k=0}^{\infty}A^{2k}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2k}, \tag{10}$$

and

$$sin_\alpha(Ax^\alpha)=\sum_{k=0}^{\infty}A^{2k+1}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2k+1)}. \tag{11}$$

Theorem 2.5 (matrix fractional Euler's formula)([25])**:** *If $0<\alpha\leq 1$, $i=\sqrt{-1}$, and A is a real matrix, then*

$$E_\alpha(iAx^\alpha)=cos_\alpha(Ax^\alpha)+isin_\alpha(Ax^\alpha). \tag{12}$$

Theorem 2.6 (matrix fractional DeMoivre's formula)([26])**:** *If $0<\alpha\leq 1$, p is an integer, and A is a real matrix, then*

$$[cos_\alpha(Ax^\alpha)+isin_\alpha(Ax^\alpha)]^{\otimes_\alpha p}=cos_\alpha(pAx^\alpha)+isin_\alpha(pAx^\alpha). \tag{13}$$

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III. MAIN RESULTS

In this section, based on a new multiplication of fractional analytic functions, we use some methods to obtain the fractional Fourier series of two types of matrix fractional functions.

Theorem 3.1: *If* $0 < \alpha \leq 1$, t *is a real number, and* A *is a matrix, then*

$$E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha cos_\alpha\big(sin_\alpha(tAx^\alpha)\big) = \sum_{k=0}^{\infty}\frac{1}{k!}cos_\alpha(ktAx^\alpha). \tag{14}$$

And

$$E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha sin_\alpha\big(sin_\alpha(tAx^\alpha)\big) = \sum_{k=0}^{\infty}\frac{1}{k!}sin_\alpha(ktAx^\alpha). \tag{15}$$

Proof Since $E_\alpha\big(E_\alpha(itAx^\alpha)\big)$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\big(E_\alpha(itAx^\alpha)\big)^{\otimes_\alpha k}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}E_\alpha(iktAx^\alpha) \quad \text{(by matrix fractional DeMoivre's formula)}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}[cos_\alpha(ktAx^\alpha) + isin_\alpha(ktAx^\alpha)] \quad \text{(by matrix fractional Euler's formula)}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}cos_\alpha(ktAx^\alpha) + i\cdot\sum_{k=0}^{\infty}\frac{1}{k!}sin_\alpha(ktAx^\alpha)\ . \tag{16}$$

And

$$E_\alpha\big(E_\alpha(itAx^\alpha)\big)$$

$$= E_\alpha\big(cos_\alpha(tAx^\alpha) + isin_\alpha(tAx^\alpha)\big)$$

$$= E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha E_\alpha\big(isin_\alpha(tAx^\alpha)\big)$$

$$= E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha \big[cos_\alpha\big(sin_\alpha(tAx^\alpha)\big) + i\cdot sin_\alpha\big(sin_\alpha(tAx^\alpha)\big)\big]$$

$$= E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha cos_\alpha\big(sin_\alpha(tAx^\alpha)\big) + i\cdot E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha sin_\alpha\big(sin_\alpha(tAx^\alpha)\big). \tag{17}$$

It follows that

$$E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha cos_\alpha\big(sin_\alpha(tAx^\alpha)\big) = \sum_{k=0}^{\infty}\frac{1}{k!}cos_\alpha(ktAx^\alpha).$$

And

$$E_\alpha\big(cos_\alpha(tAx^\alpha)\big)\otimes_\alpha sin_\alpha\big(sin_\alpha(tAx^\alpha)\big) = \sum_{k=0}^{\infty}\frac{1}{k!}sin_\alpha(ktAx^\alpha).$$ q.e.d.

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we use some techniques to find the fractional Fourier series of two types of matrix fractional functions. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. Moreover, our results are generalizations of the results in traditional calculus. In the future, we will continue to study the problems in applied mathematics and fractional differential equations by using our methods.

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